

The York Map and the Role of Non-Inertial Frames in the Geometrical View of the Gravitational Field.

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Abstract

The role of non-inertial frames in a class of models of general relativity is clarified by means of Dirac's theory of constraints. The identification of a York canonical basis allows to give the interpretation of the gauge variables as generalized inertial effects and to identify the Dirac observables of the gravitational field with generalized tidal effects. York time is the gauge variable controlling the clock synchronization convention. Differently from special relativity, the instantaneous 3-spaces are dynamically determined.

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Dirac's constraint theory [1] is the natural language to describe both gauge theories and gravitational physics. Even if the Hamiltonian approach lies at the heart of the Faddeev-Popov measure of the path integral and of the BRST method, both particle physicists and general relativists tend to prefer the configurational one due to its manifest covariance and to the possibility of avoiding to face the problem of what is *time* in a relativistic theory. However, this is an illusion, because the problem cannot be eluded when we try to establish a well-posed Cauchy problem for the field equations, in absence of which we cannot use the existence and uniqueness theorem for the solutions of partial differential equations and speak of predictability. As a consequence, the basic problem is how to separate the gauge degrees of freedom from the gauge-invariant genuine dynamical variables (the Dirac observables, DO). While the former are (modulo some restriction) completely arbitrary, the latter have to satisfy deterministic hyperbolic partial differential equations in every completely fixed gauge. Now, only the Hamiltonian formalism has a natural (even if still heuristic from the point of view of mathematical rigor) tool to implement this separation: the Shanmugadhasan canonical transformations [2] adapted to the first- and second-class constraints of the model and to the second Noether theorem underlying their existence due to the local invariances of the action functional. This type of transformations were introduced by Dirac for the electromagnetic field in a seminal paper [3], in which its DO were identified with the transverse vector potential and electric field of the radiation gauge. This work has been extended to Yang-Mills theory in Ref.[4] and applied in Ref. [5] to the classical version of the standard model of elementary particles. See Ref.[6] for a review.

However, these results *hold only in inertial frames in Minkowski space-time* and are possible because gauge transformations act on an *inner space*: the redundant gauge variables are present only to implement the manifest covariance of the model under the action of the kinematical Poincare' transformations connecting inertial frames and under local *inner* Lie groups. As a consequence, in every formulation both at the classical and quantum level the standard model of elementary particles and its extensions are a chapter of the theory of representations of the Poincare' group in inertial frames in Minkowski space-time (the non-dynamical container of the fields).

In special relativity the structure of the light-cones is an absolute non-dynamical object [7]: they are the only information (the conformal structure) given by the theory to an (either inertial or accelerated) observer in each point of her/his world-line. There is no notion of instantaneous 3-space, of spatial distance and of one-way velocity of light between two observers ¹. The light postulates say that the two-way (or round trip) velocity of light c (only one clock is needed in its definition) is constant and isotropic. For an ideal inertial observer Einstein's convention for the synchronization of distant clocks ² selects the constant

¹ By contrast in Newton physics there are distinct absolute notions of *time* and *space*, so that we can speak of absolute simultaneity and of instantaneous Euclidean 3-spaces with the associated Euclidean spatial distance notion. This non-dynamical chrono-geometrical structure is formalized in the so called Galilei space-time. The Galilei relativity principle assumes the existence of preferred inertial frames with inertial Cartesian coordinates centered on inertial observers, connected by the kinematical group of Galilei transformations. In Newton gravity the equivalence principle states the equality of inertial and gravitational mass. In non-inertial frames inertial (or fictitious) forces proportional to the mass of the body appear in Newton's equations.

² The inertial observer A sends a ray of light at x_i^o to a second accelerated observer B , who reflects it

time hyper-planes of the inertial frame having the observer as time axis as the instantaneous Euclidean 3-spaces, with their Euclidean 3-geodesic spatial distance and with the one-way and two-way velocities of light equal.

But this convention does not work for realistic accelerated observers, because coordinate singularities are produced in the attempt (the 1+3 point of view) to build (either Fermi or rotating) 4-coordinates around the observer world-line. They must use the more complex conventions arising from the introduction of an extra structure: a global 3+1 splitting of Minkowski space-time (a choice of *time*, the starting point of the Hamiltonian formalism). Each space-like leaf of the associated foliation is *both a Cauchy surface for the field equations and a convention* (different from Einstein's one) *for clock synchronization*. If we introduce *Lorentz-scalar observer-dependent radar 4-coordinates* $x^\mu \mapsto \sigma^A = (\tau; \sigma^r)$, where x^μ are Cartesian coordinates, τ is an arbitrary monotonically increasing function of the proper time of the observer and σ^r curvilinear 3-coordinates having the observer world-line as origin, this leads to the definition of a *non-inertial frame* centered on the accelerated observer [8]. Every 3+1 splitting, satisfying certain Møller restrictions (to avoid coordinate singularities) and with the leaves tending to hyper-planes at spatial infinity (so that there are asymptotic inertial observers to be identified with the fixed stars), gives a conventional definition of instantaneous 3-space (in general a Riemannian 3-manifold), of 3-geodesic spatial distance and of one-way velocity of light (in general both point-dependent and anisotropic). The inverse coordinate transformation $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ defines the *embedding* of the simultaneity surfaces Σ_τ into Minkowski space-time. The 3+1 splitting leads to the following induced 4-metric (a functional of the embedding): ${}^4g_{AB}(\tau, \sigma^r) = \frac{\partial z^\mu(\sigma)}{\partial \sigma^A} {}^4\eta_{\mu\nu} \frac{\partial z^\nu(\sigma)}{\partial \sigma^B} = {}^4g_{AB}[z(\sigma)]$.

Parametrized Minkowski theories [9], [7] allow to give a description of every isolated system (particles, strings, fields, fluids), in which the transition from a 3+1 splitting to another one (i.e. a change of clock synchronization convention) is a *gauge transformation*. Given any isolated system admitting a Lagrangian description, one makes the coupling of the system to an external gravitational field and then replaces the 4-metric ${}^4g_{\mu\nu}(x)$ with the induced metric ${}^4g_{AB}[z(\tau, \sigma^r)]$ associated to an arbitrary admissible 3+1 splitting. The Lagrangian now depends not only on the matter configurational variables but also on the embedding variables $z^\mu(\tau, \sigma^r)$ (whose conjugate canonical momenta are denoted $\rho_\mu(\tau, \sigma^r)$). Since the action principle turns out to be invariant under *frame-preserving diffeomorphisms*, at the Hamiltonian level there are four first-class constraints $\mathcal{H}_\mu(\tau, \sigma^r) = \rho_\mu(\tau, \sigma^r) - l_\mu(\tau, \sigma^r) T^{\tau\tau}(\tau, \sigma^r) - z_s^\mu(\tau, \sigma^r) T^{\tau s}(\tau, \sigma^r) \approx 0$ in strong involution with respect to Poisson brackets, $\{\mathcal{H}_\mu(\tau, \sigma^r), \mathcal{H}_\nu(\tau, \sigma_1^r)\} = 0$. Here $l_\mu(\tau, \sigma^r)$ are the covariant components of the unit normal to Σ_τ , while $z_s^\mu(\tau, \sigma^r) = \frac{\partial z^\mu(\tau, \sigma^r)}{\partial \sigma^s}$ are the components of three independent vectors tangent to Σ_τ . The quantities $T^{\tau\tau}$ and $T^{\tau s}$ are the components of the energy-momentum tensor of the matter inside Σ_τ describing its energy- and momentum- densities. As a consequence, Dirac's theory of constraints implies that the configuration variables $z^\mu(\tau, \sigma^r)$ are arbitrary *gauge variables*. Therefore, all the admissible 3+1 splittings, namely all the admissible conventions for clock synchronization, and all the admissible non-inertial frames centered on time-like observers are *gauge equivalent*.

By adding four gauge-fixing constraints $\chi^\mu(\tau, \sigma^r) = z^\mu(\tau, \sigma^r) - z_M^\mu(\tau, \sigma^r) \approx 0$ ($z_M^\mu(\tau, \sigma^r)$ being an admissible embedding), satisfying the orbit condition $\det |\{\chi^\mu(\tau, \sigma^r), \mathcal{H}_\nu(\tau, \sigma_1^r)\}| \neq 0$,

towards A. The reflected ray is reabsorbed by the inertial observer at x_f^o . The convention states that the clock of B at the reflection point must be synchronized with the clock of A when it signs $\frac{1}{2}(x_i^o + x_f^o)$.

we identify the description of the system in the associated non-inertial frame centered on a given time-like observer. The resulting effective Hamiltonian for the τ -evolution turns out to contain the potentials of the *relativistic inertial forces* present in the given non-inertial frame. As a consequence, the gauge variables $z^\mu(\tau, \sigma^r)$ describe the *spatio-temporal appearances* of the phenomena in non-inertial frames, which, in turn, are associated to *extended* physical laboratories using a metrology for their measurements compatible with the notion of simultaneity (distant clock synchronization convention) of the non-inertial frame (think to the description of the Earth given by GPS). Therefore, notwithstanding mathematics tends to use only coordinate-independent notions, physical metrology forces us to consider intrinsically coordinate-dependent quantities like the non-inertial Hamiltonians. For instance, the motion of satellites around the Earth is governed by a set of empirical coordinates contained in the software of NASA computers: this is a *metrological standard of space-time around the Earth*.

Inertial frames centered on inertial observers are a special case of gauge fixing in parametrized Minkowski theories. For each configuration of an isolated system there is an special 3+1 splitting associated to it: the foliation with space-like hyper-planes orthogonal to the conserved time-like 4-momentum of the isolated system. This identifies an intrinsic inertial frame, the *rest-frame*, centered on a suitable inertial observer (the covariant non-canonical Fokker-Pryce center of inertia of the isolated system) and allows to define the *Wigner-covariant inertial rest-frame instant form of dynamics* for every isolated system, which allows to give a new formulation of the relativistic kinematics [10] of N-body systems and continuous media (relativistic centers of mass and canonical relative variables, rotational kinematics and dynamical body frames, multipolar expansions, Møller radius) and to find the theory of relativistic orbits. Instead *non-inertial rest frames* are 3+1 splittings of Minkowski space-time having the associated simultaneity 3-surfaces tending to Wigner hyper-planes at spatial infinity.

Moreover it is now possible to define relativistic and non-relativistic quantum mechanics of particles in non-inertial frames [11], with a multi-temporal quantization scheme in which the gauge variables $z^\mu(\tau, \sigma^r)$ (the appearances) are *c*-numbers (generalized times) and only the particle degrees of freedom are quantized. What is still lacking is the quantization of a scalar field in non-inertial frames. Torre and Varadarajan [12] have shown that the traditional Tomonaga-Schwinger approach does not lead in general to a unitary evolution. From the 3+1 point of view we have to restrict the 3+1 splittings to those whose simultaneity leaves admit an instantaneous Fock space, in which the Bogoliubov transformation between two such leaves of the foliation is Hilbert-Schmidt (unitary evolution). Moreover all such admissible 3+1 splittings must be unitarily gauge equivalent.

Things change dramatically when gravity is taken into account [7]. In general relativity there is no absolute notion: the full chrono-geometrical structure of space-time is dynamical. The relativistic description of gravity abandons the relativity principle and replaces it with the equivalence principle. Special relativity can be recovered only locally by a freely falling observer in a neighborhood where tidal effects are negligible. As a consequence, *global inertial frames do not exist*.

The general covariance of Einstein's formulation of general relativity leads to a type of gauge symmetry acting also on space-time. The Hilbert action is invariant under coordinate transformations (*passive* off-shell diffeomorphisms as local Noether transformations),

whereas the abstract differential geometric formulation is invariant under *active* diffeomorphisms of the space-time 4-manifold extended to tensors (on-shell dynamical symmetries of Einstein's equations; it is assumed that the space of solutions exists according to some notion of integrability). Both in the off-shell and on-shell viewpoints the gauge group is a group of diffeomorphisms acting also on space-time (the same happens in every model with some type of reparametrization invariance).

Even if from a mathematical point of view the gauge variables are still arbitrary degrees of freedom not determined by the field equations, they are no more redundant variables of an inner space, but are connected with the *appearances* of phenomena in the various coordinate systems of Einstein's space-times.

In Einstein's geometrical view of the gravitational field the basic configuration variable is the metric tensor over space-time (10 fields), which, differently from every other field, has a double role:

- i) it is the mediator of the gravitational interaction, like every other gauge field;
- ii) it describes the dynamical chrono-geometrical structure of space-time by means of the line element $ds^2 = {}^4g_{\mu\nu}(x) dx^\mu dx^\nu$. As a consequence, it *teaches relativistic causality* to the other fields: now the conformal structure (the allowed paths of light rays) is point-dependent.

In canonical ADM metric gravity [13] (and in its tetrad gravity extension [14, 15] needed for fermions ³) we have again to start with the same pattern of 3+1 splittings, to be able to define the Cauchy and simultaneity surfaces for Einstein's equations. As a consequence, and having in mind the inclusion of particle physics, we must select a family of *non-compact* space-times M^4 with the following properties:

- i) *globally hyperbolic* and *topologically trivial*, so that they can be foliated with space-like hyper-surfaces Σ_τ diffeomorphic to R^3 (3+1 splitting of space-time with τ , the scalar parameter labeling the leaves, as a *mathematical time*);
- ii) *asymptotically flat at spatial infinity* and with boundary conditions at spatial infinity independent from the direction, so that the *spi group* of asymptotic symmetries is reduced to the Poincare' group with the ADM Poincare' charges as generators. In this way we can eliminate the *super-translations*, namely the obstruction to define angular momentum in general relativity, and we have the same type of boundary conditions which are needed to get well defined non-Abelian charges in Yang-Mills theory, opening the possibility of a unified description of the four interactions with all the fields belonging to same function space [13], [6, 7]. All these requirements imply that the *admissible foliations* of space-time must have the space-like hyper-surfaces tending in a direction-independent way to Minkowski space-like hyper-planes at spatial infinity, which moreover must be orthogonal there to the ADM 4-momentum. Therefore, M^4 is *asymptotically Minkowskian* [16] with the asymptotic Minkowski metric playing the role of an *asymptotic background*. Moreover the simultaneity 3-surfaces (the Riemannian instantaneous 3-spaces) must admit an involution (Lichnerowicz 3-manifolds [17]) allowing the definition of a generalized Fourier transform with its associated concepts of positive and negative energy, so to avoid the claimed impossibility to define particles in curved space-times.

³ This leads to an interpretation of gravity based on a congruence of time-like observers endowed with orthonormal tetrads: in each point of space-time the time-like axis is the unit 4-velocity of the observer, while the spatial axes are a (gauge) convention for observer's gyroscopes.

iv) All the fields have to belong to suitable *weighted Sobolev spaces* so that; i) the admissible space-like hyper-surfaces are Riemannian 3-manifolds without asymptotically vanishing Killing vectors [16, 18] (we furthermore assume the absence of any Killing vector); ii) the inclusion of particle physics leads to a formulation without Gribov ambiguity [19],[4].

In absence of matter the class of Christodoulou-Klainermann space-times [20], admitting asymptotic ADM Poincare' charges and an asymptotic flat metric meets these requirements.

This formulation, the *rest-frame instant form of metric and tetrad gravity*, emphasizes the role of *non-inertial frames* (the only ones existing in general relativity): each admissible 3+1 splitting identifies a global non-inertial frame centered on a time-like observer. In these space-times each simultaneity surface is the rest frame of the 3-universe, there are asymptotic inertial observers (the fixed stars) and the switching off of the Newton constant in presence of matter leads to a deparametrization of these models of general relativity to the non-inertial rest-frame instant form of the same matter with the ADM Poincare' charges collapsing into the usual kinematical Poincare' generators. This class of space-times is suitable to describe the solar system (or the galaxy), is compatible with particle physics and allows to avoid the splitting of the metric into a background one plus a perturbation ⁴. With the addition of suitable asymptotic terms it can probably be adapted to cosmology [23].

The first-class constraints of canonical gravity (8 in metric gravity, 14 in tetrad gravity ⁵) imply the existence of an equal number of arbitrary gauge variables and of only 2+2 genuine physical degrees of freedom of the gravitational field: $r_{\bar{a}}(\tau, \sigma^r)$, $\pi_{\bar{a}}(\tau, \sigma^r)$. It can be shown [13, 14, 24] that the super-hamiltonian constraint generates Hamiltonian gauge transformations implying the *gauge equivalence* of clock synchronization conventions like it happens in special relativity ⁶ (no Wheeler-DeWitt interpretation of it as a Hamiltonian). As shown in Refs.[24] ⁷, the gauge variables describe *generalized inertial effects* (the appearances), while the 2+2

⁴ This splitting is the basic tool for the linearization of Einstein's equations (see the theory of gravitational waves) and for their replacement with a non-geometric spin-two theory over Minkowski space-time, in which diffeomorphisms acting on space-time are discontinuously replaced with gauge transformations acting on an inner space. However, as shown by Deser [21], the non-geometrical spin-two theory becomes inconsistent if we add the energy-momentum tensor $T_{\mu\nu}$ of dynamical matter as a source: the only way to recover consistency (at the price of loosing an energy conservation law for the gravitational field) is to recover Einstein's theory. Notwithstanding Deser's result, particle physicists prefer to rely on Feynman's statement [22] that *the geometrical interpretation is not really necessary or essential to physics*. The basic reason seems to be the absence of an energy conservation law for the gravitational field, replaced by a coordinate-dependent notion of energy density.

⁵ Tetrad gravity has 10 primary first class constraints and 4 secondary first class ones. Six of the primary constraints describe the extra freedom in the choice of the tetrads. The other 4 primary (the vanishing of the momenta of the lapse and shift functions) and 4 secondary (the super-Hamiltonian and super-momentum constraints) constraints are the same as in metric gravity. In Ref.[14] 13 of the 14 constraints were solved: the super-Hamiltonian one can be solved only after linearization [15].

⁶ The special relativistic constraints $\mathcal{H}_\mu(\tau, \sigma^r) \approx 0$ are replaced by the super-hamiltonian and super-momentum ones.

⁷ In these papers there is also a solution of Einstein's Hole Argument: in this class of space-times it is possible to identify the point-events of space-time by means of the four tidal degrees of freedom of the gravitational field. In other words, space-time and gravitational field are two faces of the same entity. The

gauge invariant DO describe *generalized tidal effects*. In Refs.[14, 15] a Shanmugadhasan canonical basis, adapted to 13 of the 14 tetrad gravity first class constraints (not to the super-Hamiltonian one) was found. With its help it can be shown [24] that a completely fixed Hamiltonian gauge is equivalent to the choice of a *non-inertial frame* with its adapted radar coordinates centered on an accelerated observer and its instantaneous 3-spaces (simultaneity surfaces): again this corresponds to an extended physical laboratory ⁸.

In the rest-frame instant form of gravity [13, 14], due to the DeWitt surface term the effective Hamiltonian is not weakly zero (*no frozen picture* of dynamics), but is given by the weak ADM energy $E_{ADM} = \int d^3\sigma \mathcal{E}_{ADM}(\tau, \sigma^r)$ (it is the analogous of the definition of the electric charge as the volume integral of the charge density in electromagnetism). The ADM energy density depends on the gauge variables, namely it is a coordinate-dependent quantity (the *problem of energy* in general relativity). In a completely fixed gauge, in which the inertial effects are given functions of the DO, $\mathcal{E}_{ADM}(\tau, \sigma^r)$ becomes a well defined function only of the DO's and there is a deterministic evolution of the DO's (the tidal effects) given by the Hamilton equations. A universe M^4 (a 4-geometry) is the equivalence class of all the completely fixed gauges with gauge equivalent Cauchy data for the DO on the associated Cauchy and simultaneity surfaces Σ_τ . In each completely fixed gauge (an off-shell non-inertial frame determined by some set of gauge-fixing constraints determining the gauge variables in terms of the tidal ones) we find the solution for the DO in that gauge (the tidal effects) and then the explicit form of the gauge variables (the inertial effects). As a consequence, the final admissible (on-shell gauge equivalent) non-inertial frames associated to a 4-geometry (and their instantaneous 3-spaces, i.e. their clock synchronization conventions) are *dynamically determined* [24].

A first application of this formalism has been the determination [15] of *post-Minkowskian background-independent gravitational waves* in a completely fixed non-harmonic 3-orthogonal gauge with diagonal 3-metric. It can be shown that the requirements $r_{\bar{a}}(\tau, \sigma^r) \ll 1$, $\pi_{\bar{a}}(\tau, \sigma^r) \ll 1$ lead to a weak field approximation based on a Hamiltonian linearization scheme: i) linearize the Lichnerowicz equation (i.e. the super-Hamiltonian constraint), determine the conformal factor of the 3-metric and then the lapse and shift functions; ii) find E_{ADM} in this gauge and disregard all the terms more than quadratic in the DO; iii) solve the Hamilton equations for the DO. In this way we get a solution of linearized Einstein's equations, in which the configurational DO $r_{\bar{a}}(\tau, \sigma^r)$ play the role of the *two polarizations* of the gravitational wave.

In Refs.[25, 26] there is the description of relativistic fluids and of the Klein-Gordon field in the framework of parametrized Minkowski theories. This formalism allows to get the

previous identification is not valid in spatially compact space-times without boundary, where the Dirac Hamiltonian weakly vanishes and there is a frozen picture of dynamics.

⁸ Let us remark that, if we look at Minkowski space-time as a special solution of Einstein's equations with $r_{\bar{a}}(\tau, \sigma^r) = \pi_{\bar{a}}(\tau, \sigma^r) = 0$ (zero Riemann tensor, no tidal effects, only inertial effects), we find [13] that the dynamically admissible 3+1 splittings (non-inertial frames) must have the simultaneity surfaces Σ_τ *3-conformally flat*, because the conditions $r_{\bar{a}}(\tau, \sigma^r) = \pi_{\bar{a}}(\tau, \sigma^r) = 0$ imply the vanishing of the Cotton-York tensor of Σ_τ . Instead, in special relativity, considered as an autonomous theory, all the non-inertial frames compatible with the Møller conditions are admissible [8], namely there is much more freedom in the conventions for clock synchronization.

Lagrangian of these matter systems in the formulation of tetrad gravity of Refs.[14, 15]. The resulting first-class constraints depend only on the mass density $\mathcal{M}(\tau, \sigma^r)$ (which is metric-dependent) and the mass-current density $\mathcal{M}_r(\tau, \sigma^r)$ (which is metric-independent) of the matter. For Dirac fields the situation is more complicated due to the presence of second class constraints (see Ref.[27] for the case of parametrized Minkowski theories with fermions). It turns out that the point Shanmugadhasan canonical transformation of Ref.[15], adapted to 13 of the 14 first class constraints is not suited for the inclusion of matter due to its *non-locality*. Therefore the search started for a local point Shanmugadhasan transformation adapted only to 10 of the 14 constraints, i.e. not adapted to the super-Hamiltonian and super-momentum constraints.

The new insight came from the so-called York - Lichnerowicz conformal approach [28, 29, 30, 31] to metric gravity in globally hyperbolic (*but spatially compact*) space-times. The starting point is the decomposition ${}^3g_{ij} = \phi^4 {}^3\hat{g}_{ij}$ of the 3-metric on an instantaneous 3-space Σ_o of a 3+1 splitting of space-time in the product of a *conformal factor* $\phi = (det {}^3g)^{1/12}$ and a *conformal 3-metric* ${}^3\hat{g}_{ij}$ with $det {}^3\hat{g}_{ij} = 1$ (${}^3\hat{g}_{ij}$ contains 5 of the 6 degrees of freedom of ${}^3g_{ij}$). The extrinsic curvature 3-tensor ${}^3K_{ij}$ of Σ_o is decomposed in its trace 3K (the *York time*) plus the *distorsion tensor*, which is the sum of a TT⁹ symmetric 2-tensor ${}^3A_{ij}$ (2 degrees of freedom) plus the 3-tensor ${}^3W_{i;j} + {}^3W_{j;i} - \frac{2}{3} {}^3g_{ij} {}^3W^k{}_{;k}$ depending on a covariant 3-vector 3W_i (*York gravitomagnetic vector potential*; 3 degrees of freedom). Having fixed the lapse and shift functions of the 3+1 splitting and having put ${}^3K = const.$, one assigns ${}^3\hat{g}_{ij}$ and ${}^3A_{ij}$ on the Cauchy surface Σ_o . Then, 3W_i is determined by the super-momentum constraints on Σ_o and ϕ is determined by the super-Hamiltonian constraint on Σ_o . Then, the remaining Einstein's equations (see Refs.[18, 28] for the existence and unicity of solutions) determine the time derivatives of ${}^3g_{ij}$ and of ${}^3K_{ij}$, allowing to find the time development from the initial data on Σ_o . However, a canonical basis adapted the the previous splittings was never found. The only result is contained in Ref.[32], where it was shown that, having fixed 3K , the transition from the non-canonical variables ${}^3\hat{g}_{ij}$, ${}^3A_{ij}$, 3W_i to the space of the gravitational initial data satisfying the constraints is a canonical transformation, named *York map*.

In Ref.[33] a new parametrization of the original 3-metric ${}^3g_{ij}$ was proposed, which allows to find local point Shanmugadhasan canonical transformation, adapted to 10 of the 14 constraints of tetrad gravity, implementing a York map. The 3-metric ${}^3g_{rs}$ may be diagonalized with an *orthogonal* matrix $V(\theta^r)$, $V^{-1} = V^T$, $det V = 1$, depending on 3 Euler angles θ^r . The gauge Euler angles θ^r give a description of the 3-coordinate systems on Σ_τ from a local point of view, because they give the orientation of the tangents to the 3 coordinate lines through each point (their conjugate momenta $\pi_i^{(\theta)}$ are determined by the super-momentum constraints and replace the York gravitomagnetic potential 3W_i), ϕ is the conformal factor of the 3-metric, i.e. the unknown in the super-hamiltonian constraint¹⁰ (its conjugate momentum is the gauge variable describing the form of the simultaneity surfaces Σ_τ), while the two independent eigenvalues of the conformal 3-metric ${}^3\hat{g}_{rs}$ (with determinant equal to 1)

⁹ Traceless and transverse with respect to the conformal 3-metric.

¹⁰ The only role of the conformal decomposition ${}^3g_{ij} = \phi^4 {}^3\hat{g}_{ij}$ is to identify the conformal factor ϕ as the natural unknown in the super-Hamiltonian constraint, which becomes the *Lichnerowicz equation*. See Ref.[13] for a different justification of this result based on constraint theory and the two notions of strong and weak ADM energy.

describe the genuine *tidal* effects $R_{\bar{a}}, \bar{a} = 1, 2$, of general relativity (the non-linear "graviton", with conjugate momenta $\Pi_{\bar{a}}$). In the York canonical basis [33] the gauge variable, which describes the freedom in the choice of the clock synchronization convention, i.e. in the definition of the instantaneous 3-spaces Σ_τ , is the trace ${}^3K(\tau, \sigma^r)$ of the extrinsic curvature of Σ_τ . It is both the York time and the momentum conjugate to the conformal factor.

The tidal effects $R_{\bar{a}}(\tau, \sigma^r), \Pi_{\bar{a}}(\tau, \sigma^r)$, are DO *only* with respect to the gauge transformations generated by 10 of the 14 first class constraints. Let us remark that, if we fix completely the gauge and we go to Dirac brackets, then the only surviving dynamical variables $R_{\bar{a}}$ and $\Pi_{\bar{a}}$ become two pairs of *non canonical* DO for that gauge: the two pairs of canonical DO have to be found as a Darboux basis of the copy of the reduced phase space identified by the gauge and they will be (in general non-local) functionals of the $R_{\bar{a}}, \Pi_{\bar{a}}$ variables. This shows the importance of canonical bases like the York one: the tidal effects are described by *local* functions of the 3-metric and its conjugate momenta.

Since the conformal factor ϕ and the momenta $\pi_i^{(\theta)}$ conjugate to the Euler angles θ^r are determined by the super-Hamiltonian and super-momentum constraints, the *arbitrary gauge variables* of the York canonical basis are $\alpha_{(a)}, \varphi_{(a)}, \theta^i, \pi_{\bar{\phi}}, n$ and $\bar{n}_{(a)}$. As shown in Refs.[24, 33], they describe the following generalized *inertial effects*:

a) the angles $\alpha_{(a)}(\tau, \sigma^r)$ and the boost parameters $\varphi_{(a)}(\tau, \sigma^r)$ describe the arbitrariness in the choice of a tetrad to be associated to a time-like observer, whose world-line goes through the point (τ, σ^r) . They fix *the unit 4-velocity of the observer and the conventions for the gyroscopes and their transport along the world-line of the observer*.

b) the angles $\theta^i(\tau, \sigma^r)$ (depending only on the 3-metric) describe the arbitrariness in the choice of the 3-coordinates on the simultaneity surfaces Σ_τ of the chosen non-inertial frame centered on an arbitrary time-like observer. Their choice induces a pattern of *relativistic standard inertial forces* (centrifugal, Coriolis,...), whose potentials are contained in the weak ADM energy E_{ADM} . These inertial effects are the relativistic counterpart of the non-relativistic ones (they are present also in the non-inertial frames of Minkowski space-time).

c) the *shift* functions $\bar{n}_{(a)}(\tau, \sigma^r)$, appearing in the Dirac Hamiltonian, describe which points on different simultaneity surfaces have the same numerical value of the 3-coordinates. They are the inertial potentials describing the effects of the non-vanishing off-diagonal components ${}^4g_{\tau r}(\tau, \sigma^r)$ of the 4-metric, namely they are the *gravito-magnetic potentials*¹¹ responsible of effects like the dragging of inertial frames (Lens-Thirring effect) [31] in the post-Newtonian approximation.

d) $\pi_{\bar{\phi}}(\tau, \sigma^r)$, i.e. the York time ${}^3K(\tau, \sigma^r)$, describes the arbitrariness in the shape of the simultaneity surfaces Σ_τ of the non-inertial frame, namely the arbitrariness in the choice of the convention for the synchronization of distant clocks. Since this variable is present in the Dirac Hamiltonian, it is a *new inertial potential* connected to the problem of the relativistic freedom in the choice of the *instantaneous 3-space*, which has no non-relativistic analogue (in Galilei space-time time is absolute and there is an absolute notion of Euclidean 3-space). Its effects are completely unexplored. For instance, since the sign of the trace of

¹¹ In the Post-Newtonian approximation in *harmonic gauges* they are the counterpart of the electro-magnetic vector potentials describing magnetic fields [31], [15]: A) $N = 1 + n$, $n \stackrel{def}{=} -\frac{4\epsilon}{c^2} \Phi_G$ with Φ_G the *gravito-electric potential*; B) $n_r \stackrel{def}{=} \frac{2\epsilon}{c^2} A_{Gr}$ with A_{Gr} the *gravito-magnetic potential*; C) $E_{Gr} = \partial_r \Phi_G - \partial_\tau (\frac{1}{2} A_{Gr})$ (the *gravito-electric field*) and $B_{Gr} = \epsilon_{ruv} \partial_u A_{Gr} = c \Omega_{Gr}$ (the *gravito-magnetic field*). Let us remark that in arbitrary gauges the analogy with electro-magnetism breaks down.

the extrinsic curvature may change from a region to another one on the simultaneity surface Σ_τ , *the associated inertial force in the Hamilton equations may change from attractive to repulsive in different regions.*

e) the *lapse* function $N(\tau, \sigma^r) = 1 + n(\tau, \sigma^r)$, the lapse function appearing in the Dirac Hamiltonian, describes the arbitrariness in the choice of the unit of proper time in each point of the simultaneity surfaces Σ_τ , namely how these surfaces are packed in the 3+1 splitting.

As a consequence, differently from special relativity, the conventions for clock synchronization and the whole chrono-geometrical structure of M^4 (gravito-magnetism, 3-geodesic spatial distance on Σ_τ , trajectories of light rays in each point of M^4 , one-way velocity of light) are *dynamically determined* [24].

The use of Dirac theory of constraints introduces a different point of view on the gauge-fixing and the Cauchy problem. While the gauge fixing to the extra 6 primary constraints fixes the tetrads (i.e. the spatial gyroscopes and their transport law), the gauge fixing to the 4 primary plus 4 secondary constraints follows a different scheme from the one used in the York-Lichnerowicz approach, which influenced contemporary numerical gravity. Firstly one adds the 4 gauge fixings to the secondary constraints (the super-Hamiltonian and super-momentum ones), i.e. one fixes 3K , i.e. the simultaneity 3-surface, and the 3-coordinates on it (namely 3 of the 5 degrees of freedom of the conformal 3-metric ${}^3\hat{g}_{ij}$). The preservation in time of these 4 gauge fixings generates other 4 gauge fixing constraints determining the lapse and shift functions consistently with the shape of the simultaneity 3-surface and with the choice of 3-coordinates on it (here is the main difference with the conformal approach and most of the approaches to numerical gravity).

The clarification of the interpretational issues allowed by the York canonical basis will allow to face many problems.

1) To understand better the Hamiltonian distinction between inertial and tidal effects, a detailed study of the Post-Newtonian solutions of Einstein's equations adopted by the IAU conventions [34] for the barycentric and geocentric celestial reference frames has begun. In particular it will clarify the mixing of the general relativistic effects like the gravitational redshift with the special relativistic ones like the Doppler effect and the Coriolis forces near the rotating Earth. This is no more an academic research, because in a few years the European Space Agency (ESA) will start the mission ACES [35] about the synchronization of a high-precision laser-cooled atomic clock on the space station with similar clocks on the Earth surface by means of microwave signals. If the accuracy of 5 picosec. will be achieved, it will be possible to make a coordinate-dependent test of effects at the order $1/c^3$, like the second order Sagnac effect (sensible to Earth rotational acceleration) and the general relativistic Shapiro time-delay created by the geoid [36]. It will be important to find the Post-Newtonian deviation from Einstein's convention to be able to synchronize two such clocks and to understand which metrological protocols have to be used for time dissemination at this level of accuracy. Therefore, the problem of clock synchronization is becoming every day more important due to GPS, to the ACES mission of ESA, to the Bepi-Colombo mission to Mercury and to the future space navigation inside the solar system.

2) The geometric vision of space-time will soon be enriched with the Hamiltonian reformulation of the Newman-Penrose formalism, in particular of the 10 Weyl scalars. This will allow a) to search the Bergmann observables [24] (special DO describing *scalar tidal effects*) and try to understand which inertial effects may have a coordinate-independent form (like

gravito-magnetism) and which are intrinsically coordinate-dependent like the ADM energy density; b) to look for the existence of a closed Poisson algebra of scalars and for Shanmugadhasan canonical bases incorporating the Bergmann observables, to be used to find new expressions for the super-hamiltonian and super-momentum constraints, hopefully easier to be solved. If the Torre-Varadarajan no-go theorem [12] can be avoided and the scalar field can be quantized in an admissible set of non-inertial frames in Minkowski space-time, it will be possible to arrive at a multi-temporal background- and coordinate- independent multi-temporal quantization (see Ref.[11]) of the gravitational field, in which only the Bergmann observables (the scalar tidal effects) are quantized.

3) The study the 2-body problem in general relativity in various coordinate systems at least in the weak field approximation, with a Grassmann regularization of the self-energies, following the track of Refs.[37] is now possible. In these papers the use of Grassmann-valued electric charges to regularize the Coulomb self-energies allowed to arrive to the Darwin and Salpeter potentials starting from classical electrodynamics of scalar and spinning particles, instead of deriving them from quantum field theory. The solution of the Lichnerowicz equation would allow to find the expression of the relativistic Newton and gravito-magnetic action-at-a-distance potentials between the two bodies (sources, among other effects, of the Newtonian tidal effects) and the coupling of the particles to the DO of the gravitational field (the genuine tidal effects) in various radar coordinate systems: it would amount to a re-summation of the $1/c$ expansions of the Post-Newtonian approximation. Also the relativistic version of the quadrupole formula for the emission of gravitational waves from the binary system could be obtained and some understanding of how is distributed the gravitational energy in different coordinate systems could be obtained ¹². It would also be possible to study the deviations induced by Einstein's theory from the Keplerian standards for problems like the radiation curves of galaxies, whose Keplerian interpretation implies the existence of dark matter ¹³.

4) With more general types of matter (fluids, electro-magnetic field) we could define Hamiltonian numerical gravity (for instance with a post-Minkowskian development in powers of the Newton constant) and try to find strong-field approximations to be used in the gravitational collapse and to find the strong-field deviations from the Newton potential. This last problem is completely open in every approach.

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¹² If the ADM energy-momentum pseudo- tensor will be identified, its reformulation as the energy-momentum tensor of a viscous pseudo-fluid will allow to check whether the pressure field is positive-definite or not, namely whether the gravitational energy contributes to dark energy.

¹³ Since the relativistic inertial forces are present in the Hamilton equations for the gravitational DO and for the matter in the gauge-dependent instantaneous 3-space, they may be a relativistic alternative to the MOND model [38] (modification of the non-relativistic Newton equations on the acceleration side for slow accelerations). At least part of the dark matter could be explained by relativistic inertial effects. See Ref.[39] for a possible gravito-magnetic origin of dark matter

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